Problem 3

What is your physical interpretation of the problem?

$$\text{PDE} \qquad u_t = \alpha^2 u_{xx} \qquad 0 < x < 1 \qquad 0 < t < \infty$$

$$\text{BCs} \qquad \begin{cases} u_x(0,t) = 0 \\ u_x(1,t) = 0 \end{cases} \qquad 0 < t < \infty.$$

$$\text{IC} \qquad u(x,0) = \sin(\pi x) \qquad 0 \le x \le 1$$

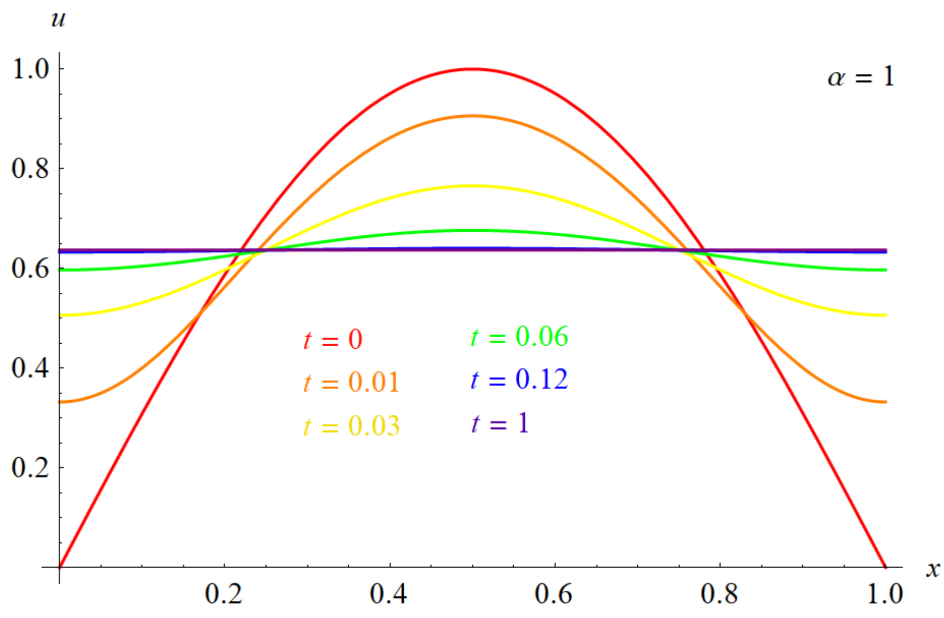
Can you draw rough sketches of this solution for various values of time? What about the steady-state temperature?

Solution

This IBVP models the temperature in a homogeneous one-dimensional rod whose lateral side is insulated; both ends of the rod ($x = 0$ and $x = 1$) are thermally insulated. The initial temperature profile is given by $u = \sin(\pi x)$. Using the method of separation of variables, the solution to the IBVP is found to be

$$u(x,t) = \frac{2}{\pi} - \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{1 + (-1)^n}{n^2 - 1} e^{-\alpha^2 n^2 \pi^2 t} \cos(n\pi x).$$

Plot this function versus x at several times for $\alpha = 1$ to illustrate the behavior of this solution.



Notice that the temperature profile is $u = \sin(\pi x)$ at $t = 0$, but over time it approaches the steady state $u = 2/\pi$. The slope of the temperature curve is zero at $x = 0$ and $x = 1$, consistent with the boundary conditions.